THE PROBABLE DIAMETERS OF THE STARS

By Henry Norris Russell

1. It would be very easy to determine the angular diameters of the stars if only we knew their surface brightness. If d is the apparent diameter of any star, m its visual magnitude, and J its surface brightness, it follows from elementary considerations that

$$d = C\ 10^{-0.2m}\ J^{-\frac{1}{2}} \qquad (1)$$

On inserting the data for the Sun ($d = 1920''$, $m = -26.72$) the constant C is found to be $0''.0087$, provided that the Sun's surface brightness is taken as unity. If $j = -2.5 \log J$[1] (so that it represents the change in stellar magnitude corresponding to a change in surface brightness in the ratio $J:1$), (1) may then be written

$$d = 0''.0087\ (0.631)^{m-j} \qquad (2)$$

No direct way of finding j is at present available; but several indirect ways are open. In comparing their results, and applying them to the stars in general, (which the present paper attempts to do in a provisional and tentative fashion), the following proposition is fundamental.

The difference of surface brightness of two stars (when expressed in stellar magnitudes) is proportional to the difference of their color indices. If i denotes the color-index we have then, for any two stars

$$j_2 - j_1 = k\ (i_2 - i_1) \qquad (3)$$

The constant k depends on the wave-lengths of the light used in measuring the surface brightness and color index, but is the same for all stars.

2. For an ideal radiator, equation (3) will be very nearly true. If the effective wave-length is λ for visual magnitudes, and λ^1 for photographic, the relation

$$k = \frac{\lambda^1}{\lambda - \lambda^1} \qquad (4)$$

will be substantially exact, so long as Wien's formula may safely be employed. At higher temperatures, when Planck's formula must be used, the value of k gradually increases, since, for an infinite

[1] In this discussion, all logarithms are supposed to be to the base 10.

temperature, this formula gives an infinite surface brightness, but a finite color index.

The only celestial body which permits of a direct observational test of (3) is the Sun. The work of Schwarzschild and Abbot makes it probable that different parts of the Sun's disk differ in color and brightness because at the center we see down deeper, and into hotter layers, than near the limb, where our line of sight is oblique. This theory has recently been strongly confirmed by Lindblad[2] who has shown that it accounts quantitatively for the distribution of energy over the disk in the different wave-lengths.

It is therefore reasonable to suppose that, on passing from the center to the limb of the Sun, we meet successively with radiative conditions very similar to those met with in the photospheres of cooler stars. The principal difference between the two cases is that the outermost layers are at the same temperature for all points of the Sun's disk, but are cooler in the cooler stars; but as these upper layers contribute very little to the general emission, the effect of this difference on the color index will presumably be small—tho it will be otherwise in the case of the line-absorption, which is mainly produced in the upper layers.

The range in brightness and color is relatively small, but this is compensated by the great accuracy of the observations. The results derived from Abbot's observations[3] of 1913 are given in Table I. The upper part of the table gives the differences of surface brightness, expressed in stellar magnitudes, between the center of the disk and regions at various distances from it, for seven wave-lengths. Color indices, relative to the center of the disk, are then formed by subtracting the brightness for 0.5955μ from the others, and the lower part of the table gives the ratio of the color index for each of the other wave-lengths to the surface brightness for the standard wave-length. These ratios are very nearly the same for all points on the disk, so that equation (3) is confirmed. For brevity the data for some of the intermediate points have been omitted. These also agree with (3).

[2] *Upsala Universitets Arsskrift*, 1920, **1**, 1340.
[3] *Smithsonian Misc. Collections*, **66**, No. 5, p. 7, 1916.

TABLE I

SURFACE BRIGHTNESS j OF THE SUN'S DISK, REFERRED TO THE CENTER

DISTANCE FROM CENTER	WAVE-LENGTH						
	0.3737μ	0.4265	0.5062	0.5955	0.6702	0.8580	1.0080
0.40	0.074	0.071	0.054	0.046	0.037	0.031	0.028
0.65	0.227	0.226	0.174	0.144	0.123	0.095	0.087
0.825	0.465	0.460	0.357	0.292	0.250	0.195	0.176
0.92	0.754	0.729	0.571	0.466	0.400	0.308	0.280
0.95	0.912	0.879	0.692	0.565	0.485	0.372	0.337

RATIO i/j

0.40	0.58	0.52	0.17	.....	−0.19	−0.31	−0.40
0.65	0.58	0.57	0.21	.....	−0.15	−0.34	−0.40
0.825	0.59	0.58	0.22	.....	−0.14	−0.33	−0.40
0.92	0.62	0.56	0.22	.....	−0.14	−0.34	−0.40
0.95	0.61	0.57	0.23	.....	−0.14	−0.34	−0.40
Means...	0.596	0.560	0.210	.....	−0.152	−0.332	−0.400
Equation (4)....	0.594	0.396	0.194	.....	−0.111	−0.306	−0.409

The mean observed values of $\frac{1}{k}$ are given at the bottom of the table. In some cases they agree closely with those predicted by equation (4) taking $\lambda = 0.5955$; but in others the discordance is considerable, indicating a departure from black body conditions.

3. It is still somewhat uncertain how far it is safe to go in assuming that similar relations between surface-brightness and color index hold good among the stars generally. There is conclusive evidence, mainly derived from variable stars, that redness and low surface brightness *generally* go together. On the other hand, Campbell[4] points out that the companion of *Sirius*, tho of spectrum A, probably has a very much smaller surface brightness than the principal star. The same argument applies, with even greater force, to the faint double companion of o^2 *Eridani*. It must remain for the future to determine whether the surface brightness of these stars is actually low, in spite of their A-type spectra, whether (as Hertz-

[4] *Pub. Astron. Soc. Pacific*, **32**, 199, 1920.

sprung[5] has suggested in another case) their light comes mainly from small areas upon their surfaces, or whether still another solution of the riddle is the true one.

Plaskett[6] has also called attention to certain eclipsing binaries in which a large difference in surface brightness is accompanied by very little difference in spectral type. In the most striking of these, RS *Vulpeculae*, the spectrum of the component of low surface brightness is barely perceptible in the violet, tho the visual brightness of the components is almost equal. This suggests strongly that there is a considerable difference in color index, in the usual direction and reduces the case to one in which the color indices of stars of nearly similar spectral type are different.

Such differences are frequent among the redder stars, and are apparently closely associated with the absolute magnitude[7]. In extreme cases they may be as great as half a magnitude, but their average value must be much smaller.

It may be concluded, provisionally, that the surface brightness of a star may be estimated, from its spectral type, with an uncertainty which is usually much less than the average difference in surface brightness between consecutive spectral types, tho it may occasionally reach this limit, and, for a very few exceptional stars, may be more seriously in error.

4. We may now proceed to discuss the various means at present available for determining the constant k, which expresses the ratio of surface brightness to color index. In order to have this really constant we must define our system of color indices. Determinations by different observers and methods differ—some being on a much more open scale than others—but all become nearly consistent when the color indices of each series are divided by that found for stars of Class K, which is called the color equation of the given system[8]. A system for which the color index of Class K is exactly 1.00 was suggested by Pickering as a standard. Let K be the value of k, referred to such a system. For any other system of color indices, with a color equation E we have

$$k = K/E$$

This new constant K is the difference of surface brightness in visual magnitudes between stars of Class A and Class K.

[5] *Astrophysical Journal*, **42**, 117, 1915.
[6] *Publ. Astron. Soc. Pacific*, **32**, 230, 1920.
[7] Seares. *Proc. Nat. Acad. Sci.*. **5**. 232. 1919
[8] See for example Schwarzschild, *Göttingen Aktinometrie;* also *H. A.*, **76**, No. 4, (1914), **80** No. 9, 1917.

5. The various investigations based on "black body" assumptions, can be utilized to determine K, if the wave-lengths which correspond to the various systems of magnitudes are known. Much less evidence is available on this important matter than there should be; but, fortunately, Parkhurst[9] in the Yerkes Actinometry, has given data which (amplified by information which he very kindly furnished by letter) makes it possible to determine[10] that the mean effective wave-length (Crova wave-length) for his photo-visual magnitudes is approximately 0.541μ and that for his photographic magnitudes is 0.428μ.

From these wave-lengths it follows from (4) that $k=3.8$ for Parkhurst's color indices. These require correction for the systematic difference between his estimates of spectral type and those made at Harvard[11] but for Class K this correction is almost negligible, and we may take $E=1.27$. Hence we find from Parkhurst's data $K=4.8$. By comparison of the Harvard systems of magnitude with Parkhurst's, it appears that the effective wave-length for the visual observations is 0.516μ and for the photographic 0.419μ.

An independent system of color indices has been derived by Rosenberg[12] from photographic spectra and gives the intensity at 0.400μ relative to that at 0.500μ. Arranging his data according to the Harvard classification of spectra, we find a mean color index of $+0.10$ for 14 stars of Class A and of $+1.83$ for 12 stars of Class K. From this, we have $K=6.7$ for visual light of wave-length 0.52μ.

We may also utilize the stellar temperatures determined spectrophotometrically by Wilsing and Scheiner[13]. These give a mean temperature of 10600° for 17 stars of Class A, and of 4400° for 28 stars of Class K, whence by Planck's formula, for $\lambda=0.52\mu$, $K=4.0$.

6. The value of K may also be determined from other astrophysical data, without the need of assuming black body conditions. The most direct evidence is afforded by the eclipsing variable U *Cephei*, in which the brighter component is of Class A and the fainter star, which eclipses it totally, of Class K[14]. The ratio of surface brightness of the two stars according to Shapley[15] is 20, according to the solution in which the star-disks are assumed to be of uniform brightness, and 17, if the disks are assumed to be dark-

[9] Ap. J. **36**, 169, 1912.
[10] Russell, *Pub. Amer. Astron. Soc.*, **2**, 162, 1915.
[11] *Ap. J.*, **36**, 188, 1912.
[12] *A. N.*, **193**, 357, 1913.
[13] *Potsdam Publ.*, **19**, 63-65, 1909.
[14] Miss Cannon, *Pop. Astron.*, **25**, 314, 1917.
[15] *Princeton Contributions*, **3**, 83, 1915.

ened at the limb. The mean of the corresponding differences of magnitude gives $K=3.2$.

A similar, but independent method depends on the determination of the differences of color index and surface brightness among totally eclipsing variables. The results for seven such systems[16] give the mean value $k=0.18$, but there is considerable uncertainty regarding the color equation of both the visual and photographic magnitudes for stars as faint as these are at minimum (from 9.2 to 12.1 on the Harvard visual scale). From the data given by Pickering[17], it appears that, when magnitudes derived from ordinary star images on the Harvard photographs are compared with the Harvard visual magnitudes, the color equation, E, between Classes A and K is +0.92 for stars brighter than 8.5 (visual); +0.75 for stars between 8.5 and 9.5, and +0.55 between 9.5 and 10.5. For stars fainter than 10.5 the very small value +0.19 was found; but this is probably affected by the well-known statistical error arising from the fact that this limiting group consists largely of stars for which the visual magnitudes have been estimated too faint. If we neglect this, and assume that the value for stars of the 10th magnitude is applicable to our stars, which at minimum average 11.2, we have $K=0.55/0.18=3.1$; but this value is unfortunately very uncertain. Re-observation of these and similar stars, by methods of accurately determined color equation, would yield a very valuable determination of this constant.

Still another method involving eclipsing binaries, but independent of the others, is the comparison of the densities of the stars of the various spectral classes, derived from such systems, with the relation between density and surface brightness, derived from visual double stars. The results of the writer's investigation[18] may be summarized as follows:

SPECTRUM	B	A	F	G	M
j	−1.2	−0.0	+0.9	+2.0	>+4.5
i	−0.3	0.0	+0.3	+0.7	+1.6

It may be concluded that the surface brightness j is 3.1 times the color index i on the Harvard scale. As the latter is +1.12 for Class K this gives $K=3.5$.

Finally, the concomitant variations in magnitude and color index among Cepheid variables may be interpreted as giving the constant

[16] Russell, Fowler and Borton, *Ap. J.*, **45**, 344, 1917.
[17] *H. A.*, **80**, 148, 1917.
[18] *Pop. Astron.*, **22**, 339, 1914.

k, provided that it is assumed that the observed variation is due to changes of temperature in a radiating surface of constant area. The last part of this assumption is very dubious but, taking it for what it is worth, we find (utilizing a table given by Seares and Shapley[19]) that for 10 stars the mean range in visual magnitude is 0.75 and in color index 0.40; this gives $k=1.87$, and, if $E=1.2$ (as assumed by the observers), $K=2.3$.

7. Collecting the results of these estimates, we have

Parkhurst, color indices	$K=4.8$
Rosenberg, color indices	6.7
Wilsing and Scheiner, temperatures	4.0
U *Cephei*, direct observations	3.2
Eclipsing variables, color index	3.1
Comparison of Ecl. Var. and binary stars	3.5
Color change of Cepheids	2.3
General mean	3.9 ± 0.4

These results are doubtless of very different accuracy; and, excepting the single instance of U *Cephei*, all of them are subject to sources of uncertainty, observational or theoretical. On the whole, they agree surprisingly well. The probable error of the mean has been computed from the residuals according to the usual rule, but should not be taken too seriously. It appears, however, that if the round number $K=4.0$ is adopted, the error is not likely to exceed one magnitude.

8. It remains to estimate the surface brightness for other spectral classes. With the value of K just adopted this will be four times the color index on the standard scale (with color equation 1.0). This scale has been discussed by Pickering[20], who finds that the results of various observers using different methods are on the whole remarkably consistent. His final table suffers, however, from the disadvantage that the color indices for intermediate spectral classes (such as F5) have been determined by interpolation between the nearest even classes. There is no more theoretical justification for assuming that the color index of Class F5 is half way between those of F0 and G0 than for assuming that G is half way between F and K. It seemed therefore desirable to repeat the computation, keeping the stars of each spectral subdivision separate. This made it necessary to exclude the magnitudes of the Draper Catalogue and the C. P. D. from the discussion, since the data given for them in *H. A.* **64** are insufficiently itemized. The

[19] *Ap. J.*, **48**, 238, 1918.
[20] *H. A.*, **80**, 147-152, 1917.

results for the other four observers are summarized in Table II[21], which gives for each observer the color index, reduced to the standard scale by division by the color equation given at the bottom of the table. Parkhurst's data have been corrected for the systematic deviation of his spectral estimates from the Harvard scale, and Schwarzschild's values depending upon the more accurate spectral classification of *H. A.* **56** have been adopted. The concluded mean scale of color indices is given in the sixth column. The average deviation from the values given by Pickering is only $\pm 0^m.03$ for spectra from B2 to G5 inclusive; but for spectra later than *K* the differences are large and systematic, averaging $+0^m.19$. The main reason for this is that the group called "M" by Pickering is composed of Classes K5 and M in nearly equal numbers, and should have been called K7 or K8. Neglect of this makes Pickering's adopted color indices for these stars too small. There is additional evidence, however, that K2 and K5 are really nearer to M than their designations indicate. The color-indices are expressed in hundredths of a magnitude.

TABLE II

SPECTRUM	KING	SCHWARZSCHILD	PARKHURST	*H. A.* **80**	MEAN
B 2	− 25	− 14	.....	− 22	− 21
B 5	− 15	− 8	− 28	− 1	− 13
B 8	− 11	− 8	− 13	− 7	− 10
A 0	0	0	0	0	0
A 2	+ 11	+ 6	+ 9	+ 14	+ 10
A 5	+ 18	+ 23	+ 17	+ 22	+ 20
F 0	+ 30	+ 33	+ 26	+ 24	+ 28
F 5	+ 40	+ 30	+ 39	+ 39	+ 37
F 8	+ 50	.....	+ 51	+ 49	+ 50
G 0	+ 63	+ 49	+ 55	+ 43	+ 53
G 5	+ 87	+ 69	+ 75	+ 71	+ 76
K 0	+100	+100	+100	+100	+100
K 2	+134	+139	+110	+116	+125
K 5	+144	+147	+121	+134	+136
M a	+147	+152	+136	+163	+145
M b	+168			+160	+154
N	.....	.....	+210	.....	+210
Color Equation	+112	101	127	89	.....

[21]Data for Class N are added from Parkhurst, *Ap. J.*, **35**, 132, 1912.

9. We are now in a position to estimate the apparent angular diameter of any star, using equation (2). The calculation may be most conveniently made by means of the two small tables given below. Table III gives the quantity $-j$ (which is -4 times the color index, on the standard scale, referred to Class G0 as standard.) The quantity taken from this table, with the spectral class as argument, is to be added algebraically to the visual magnitude of the star. Entering Table IV with this sum as argument, the diameter d of the star in thousandths of a second of arc is immediately obtained, if m-j lies between 0 and 5. For other values of this quantity it is to be remembered that a change of five magnitudes corresponds to a change in diameter by a factor of 10.

TABLE III

SPECTRUM	$-j$
B 0	$+3^m.2$
B 2	+3 .0
B 5	+2 .7
B 8	+2 .5
A 0	+2 .1
A 2	+1 .7
A 5	+1 .3
F 0	+1 .0
F 5	+0 .6
F 8	+0 .1
G 0	0 .0
G 5	−0 .9
K 0	−1 .9
K 2	−2 .9
K 5	−3 .3
M a	−3 .7
M b	−4 .0
N	−6 .3

TABLE IV

$m-j$	d	D
$0^m.0$	8″.7	9.2
0 .5	6 .9	7.3
1 .0	5 .5	5.8
1 .5	4 .3	4.6
2 .0	3 .4	3.7
2 .5	2 .7	2.9
3 .0	2 .2	2.3
3 .5	1 .7	1.8
4 .0	1 .4	1.5
4 .5	1 .1	1.2
5 .0	0 .9	0.9

10. When the parallax is known, the linear diameter of the star may also be determined. If M is the absolute magnitude of the star (the Sun's absolute magnitude being $+4.83$), and K its actual diameter, taking the Sun's diameter as unit, we find at once

$$D=9.22\ (0.631)^{M\text{-}j}$$

The values of this quantity may be taken directly from the last column of Table IV, with M-j as argument. As numerical examples we may take: (1) *Sirius*. Spectrum A, $m=-1.6$, $M=+1.3$. Hence $-j=+2.1$, $d=0''.0068$, and $D=1.9$ times the Sun's diameter. (2) α *Scorpii*. Spectrum Map, $m=+1.2$, $M=-4.0$ (with the parallax $0''.0089$ resulting from its obvious connection with Kapteyn's *Scorpius-Centaurus* group). In this case $j=-3.7$,

$d = 0''.027$, and $D = 300$ times the Sun's diameter. The actual diameter comes out remarkably large, almost equal to that of the orbit of *Mars*[22].

11. In seeking stars which have probably a large angular diameter, redness is even a more important guide than apparent brightness. If, for example, we want stars of diameter exceeding $0''.01$ we must have $m\text{-}j < -0.3$. No star of the solar type or "earlier" passes this test,—even *Sirius*, *Canopus* and *α Centauri* fail. But the limiting magnitude for Class K is +1.6, for K5, +3.0, and for Class M, +3.4, while for Class N it is +6.0,—including a number of stars in each case. A list of the stars admitted by this criterion is given in Table V.

The most remarkable case of all is the very red star VX *Andromedae* (+43° 53) which has a color index of +4.56, according to Parkhurst. The corresponding value of $-j$ would be -12.3. The visual magnitude is irregularly variable thru a small range, but averages about 8.2. This makes the computed angular diameter $0''.054$, which is greater than that of any other star in the sky. It would be straining our assumptions beyond reasonable limits to take this calculation very seriously; but it is clear none the less that the very red stars of which this is an example [23] are objects of much interest.

TABLE V

Star	Mag.	Sp	d
α Orionis	0.9	Ma	$0''.031$
α Scorpii	1.2	Map	.028
γ Crucis	1.6	Mb	.026
α Tauri	1.1	K5	.024
β Gruis	2.2	Mb	.020
α Bootis	0.2	Ko	.019
α_2 Centauri	1.7	K5	.018
β Pegasi	2.6	Mb	.016
β Androm.	2.4	Ma	.015
α Triang. Aust.	1.8	K2	.015
19 Piscium	5.3	N	.014
α Ceti	2.8	Ma	.013
η Sagittarii	3.2	Mb	.013
B Can. Venat	5.5	N	.013
β Geminorum	1.2	Ko	.012
α Arietis	2.2	K2	.012
α Hydrae	2.2	K2	.012
π Puppis	2.7	K5	.012
δ Ophiuchi	3.0	Ma	.012
σ Librae	3.4	Mb	.012

[22]Compare the calculations of Hnatek, *A. N.* **198**, 35, 1914.
[23]See Miss Cannon, *H. A.* **91**, 11, 1918.

U Antliae	5.7	N	.012
$-57^\circ.513$	5.7	N	.012
γ Draconis	2.4	K5	.011
γ Hydri	3.2	Ma	.011
μ Geminorum	3.2	Ma	.011
α Herculis	3.3	Mb	.011
D. M. $-42^\circ.2818$	6.0	N	.010
α_1 Centauri	0.3	G0	0.008
α Can. Min	0.5	F5	0.005
α Carinae	−0.9	F0	0.009
α Aquilae	0.9	A5	0.003
α Can. Maj	−1.6	A0	0.007
α Lyrae	0.1	A0	0.003
β Orionis	0.3	B8	0.0023
β Centauri	0.9	B1	0.0014
ϵ Orionis	1.7	B0	0.0009

A few of the brightest stars of "earlier" spectral classes have been added at the end of the table to illustrate the contrast in diameter between these and the redder stars.

In the present state of our knowledge, the star-diameters given above should be regarded only as rough estimates; but in most cases they probably indicate the order of magnitude of the actual diameters. The error is likely to be greatest for those stars which differ most from the solar type of spectrum; and this would be especially true of the faint red dwarf stars, among which, as Hertzsprung has shown, the effective wave-length remains nearly the same over a considerable range of absolute magnitude.

Taking the brighter component of Krüger 60 as an example, we have spectrum, Mb.; abs. mag., +11.7; and mass, 0.3 times the Sun's. The tables give $D=0.27$, and a density 15 times that of the Sun. A decrease in the estimated surface brightness by one magnitude would give $D=0.42$, density = 4, which appears more probable.

It is probably safe to conclude that a few of the brightest and reddest stars have apparent diameters of $0''.02$, or a little more, and that they are very attractive objects for investigation with the new Michelson interferometer, which has had such remarkable success in resolving *Capella* as a double star.

Princeton University Observatory
1920, October 7.